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Optical geometries and related structures *

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Abstract

Two natural optical geometries on the space $\mathcal{P}$ of all null directions over a four-dimensional Lorentzian manifold $\mathcal{M}$ are defined and studied. One of this geometries is never integrable and the other is integrable iff the metric of $\mathcal{M}$ is conformally flat. Sections of $\mathcal{P}$ forming a zero set of integrability conditions for the latter optical geometry are interpreted as principal null directions on $\mathcal{M}$.

Certain well-defined conditions on $\mathcal{P}$ are shown to be equivalent to the vanishing of the traceless part of the Ricci tensor of $\mathcal{M}$. Sections of $\mathcal{P}$ forming a zero set for these new conditions correspond to the eigendirections of the Ricci tensor of $\mathcal{M}$.

An analogy between optical and Hermitian geometries is discussed. Existing (or possible to exist) mutual counterparts between facts from optical and Hermitian geometries are listed. In this analogy, construction of the optical geometries on $\mathcal{P}$ constitutes a Lorentzian counterpart of the Atiyah–Hitchin–Singer construction of two natural almost Hermitian structures on the twistor space of four-dimensional Euclidean manifold.

Keywords: Optical geometries; Hermitian geometries; Lorentzian analog of twistor bundle
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1. Introduction

In 1922, Cartan [4] observed that in any conformally nonflat Lorentzian 4-manifold there exists at most four preferred by geometry null directions. Any such direction is now called a principal null direction and is generated by a nonvanishing vector field k_μ satisfying

$$k_{[\alpha}C_{\mu]\nu\rho[\sigma}k_{\beta]}k^\nu k^\rho = 0, \qquad (1)$$

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where $C^{\rho}_{\mu\nu\sigma}$ is the Weyl tensor. Cartan's observation was then independently noted by a number of authors. In particular, Penrose [22] found principal null directions reformulating General Relativity in terms of spinors.

An efficient method of finding principal null directions on a four-dimensional manifold $\mathcal{M}$ equipped with a Lorentzian conformally nonflat metric g uses Newman–Penrose formalism. This associates a null tetrad $(k, l, m, \bar{m})$ with g in such a way that $\Psi_4 \neq 0$, where Ψ_i $(i = 0, 1, 2, 3, 4)$ denote Weyl scalars. A principal null direction is then generated by a null vector field

$$k(z) = k + z\bar{z}l + \bar{z}m + z\bar{m}, \tag{2}$$

with a complex function z on $\mathcal{M}$ being any solution to the equation

$$\Psi_0(z) = \Psi_0 + 4z\Psi_1 + 6z^2\Psi_2 + 4z^3\Psi_3 + z^4\Psi_4 = 0. \tag{3}$$

Since Cartan's discovery there was a growing interest in studying null objects. The simplest of them, a null congruence in an oriented space–time, defines quite a rich structure – 'optical geometry' of Trautman [36]. This notion, defined previously only in four-dimensions, is now generalized to even-dimensional Lorentzian manifolds of higher dimension [12]. Higher-dimensional optical geometries appear naturally in the twistor theory. One finds [37] that there is a precise way of defining two optical geometries $\mathcal{O}_\pm$ on Penrose's [23] six-dimensional space of all null directions over four-dimensional Lorentzian manifold. These optical geometries are studied in this paper (Sections 2, 4 and 5). Their integrability properties correspond to certain properties of the metric of space–time. Our studies of $\mathcal{O}_+$ form an optical geometric reformulation of facts on natural objects on space of null directions known to Penrose [24] (see also a note about this in Refs. [18,34]).

Studying integrability conditions of $\mathcal{O}_+$ we find, in Section 5, that slightly modified conditions encode the traceless part of the Einstein equations for the space–time. These new conditions are interpreted geometrically in terms of a 1-parameter family of five-dimensional CR-structures that can be identified with the space of null directions.

Section 6 presents a way of understanding optical geometries as objects that are analogous to almost Hermitian structures. We develop Trautman's proposal [38] of inspecting known facts in both Hermitian and optical sectors, and finding their mutual counterparts from the point of view of this analogy. Finally, we present a few open problems related to Trautman's proposal.

We use the following notations and conventions. If $\mathcal{M}$ denotes a manifold then $\mathrm{T}\mathcal{M}$ (respectively, $\mathrm{T}^*\mathcal{M}$) denotes its tangent (respectively, cotangent) bundle; $\mathrm{T}_x\mathcal{M}$ (respectively, $\mathrm{T}^*_x\mathcal{M}$) denotes tangent (respectively, cotangent) space at $x \in \mathcal{M}$. If $\mathcal{K}$, $\mathcal{L}$ denote vector bundles over $\mathcal{M}$, then $\Gamma(\mathcal{K})$ denotes the set of all sections of $\mathcal{K}$, and $[\Gamma(\mathcal{K}), \Gamma(\mathcal{L})]$ denotes a set consisting of all commutators of the form $[k, l]$, where k, l are sections of $\mathcal{K}$, $\mathcal{L}$, respectively.

If $\phi : \mathcal{M} \to \mathcal{M}'$ is a diffeomorphism between two manifolds $\mathcal{M}$ and $\mathcal{M}'$, then ϕ_* denotes the transport of contravariant tensor fields from $\mathcal{M}$ to $\mathcal{M}'$. Transport of covariant tensor fields from $\mathcal{M}'$ to $\mathcal{M}$ is denoted by ϕ^*. Transport of tensor fields of general type is denoted by $\tilde{\phi}$.

A metric with signature $(++\cdots+)$ (respectively $(++\cdots+-)$) is called Euclidean (respectively, Lorentzian). In the Lorentzian case we apply Newman–Penrose formalism as presented in Ref. [11, pp. 82–87]. [1] Note that Eqs. (2) and (3) are in agreement with this convention.

2. Natural optical geometries on twistor bundle

Let $\mathcal{M}$ be a four-dimensional oriented manifold equipped with a Lorentzian metric g. Consider set $\mathbf{S}_x$ of all null directions outgoing from a given point $x \in \mathcal{M}$. This set is topologically a sphere – celestial sphere of an observer situated at x. The points of this sphere can be parametrized by a complex number z belonging to the Argand plane $\mathbb{C} \cup \{\infty\}$. A direction associated with $z \neq \infty$ is generated by a vector $k(z)$ of (2); with $z = \infty$ we associate a direction generated by vector l. Conversely, any null direction from x is either parallel to vector l or can be represented by only one such null vector $k(z)$ that $g(k(z), l) = -1$. It follows that such $k(z)$ has necessarily form (2), and defines certain $z \in \mathbb{C}$. If a direction is parallel to l we associate with it $z = \infty$.

We define a fiber bundle $\mathcal{P} = \bigcup_{x \in \mathcal{M}} \mathbf{S}_x$ over $\mathcal{M}$, so that two-dimensional spheres $\mathbf{S}_x$ are its fibers. Canonical projection $\pi : \mathcal{P} \to \mathcal{M}$ is defined by $\pi(\mathbf{S}_x) = x$. Note that any complex function $z = z(x)$ on $\mathcal{M}$ defines, via our parametrization of $\mathbf{S}_x$ by a point of the Argand plane, a section $z : \mathcal{M} \ni x \mapsto (x, z(x)) \in \mathcal{P}$ of bundle $\mathcal{P}$. It corresponds to certain field of null directions. From now on we identify sections of bundle $\mathcal{P}$ with fields of null directions on $\mathcal{M}$. I will call bundle $\mathcal{P}$ as "Penrose's twistor space", or 'twistor bundle'. It can naturally be endowed with a structure of optical geometry of Trautman [36]. Such geometry is defined on any $2m$-dimensional ($m \geq 2$) manifold $\mathcal{X}$ equipped with Lorentzian metric g by:

(1) choosing a null congruence on $\mathcal{X}$, which defines a vector bundle $\mathcal{K}$ over $\mathcal{X}$ with null fibers of dimension 1,

(2) defining an almost complex structure $\mathcal{J}$ in the vector bundle $\mathcal{H} = \mathcal{K}^{\perp}/\mathcal{K}$ in such a way, that $\mathcal{J}$ is orthogonal with respect to the metric g' induced in $\mathcal{H}$ by g.

We endow $\mathcal{P}$ with optical geometry in four steps [37]. They use the Levi-Civita connection associated with g on $\mathcal{M}$ to define objects on $\mathcal{P}$.

Step 1: Horizontal space in $\mathcal{P}$.

Consider a vector w at a point $x \in \mathcal{M}$. Let $\gamma(t)$, such that $\gamma(0) = x$, be any tangent curve to w. Let $p \in \mathcal{P}$ be a point from a fiber over x (i.e. $\pi(p) = x$). A horizontal lift of w to a vector $\tilde{w}$ at p is defined as follows.

By definition point p describes a null direction outgoing from x. We now use a parallel propagator associated with the Levi-Civita connection of g to transport this direction from x along $\gamma(t)$. In this way we obtain 1-parameter family of directions $p(t)$. Due to the properties of the parallel transport these new directions are null. Hence $p(t)$ can be interpreted as a

[1] A Euclidean analog of this formalism is presented in [20].

curve in $\mathcal{P}$ such that $p(0) = p$. A horizontal lift of w is such a vector $\tilde{w}$ among vectors tangent at p to this curve that $\pi_*(\tilde{w}) = w$. It turns out that definition of $\tilde{w}$ does not depend on the choice of the tangent curve $\gamma(t)$ at x. It follows that horizontal lifts of linearly independent vectors at x are linearly independent at p. Hence the whole $\mathrm{T}_x\mathcal{M}$ at x can be lifted to some four-dimensional vector space H_p at p. H_p is called a horizontal space at $p \in \mathcal{P}$, and its two-dimensional complement V_p to $\mathrm{T}_p\mathcal{P}$ is called a vertical space. It follows that V_p is also tangent to the fiber $\mathbf{S}_x$ at the point p. In this way for any point $p \in \mathcal{P}$ we have a natural splitting of its tangent space into direct sum $\mathrm{T}_p\mathcal{P} = V_p \bigoplus H_p$. Moreover, V_p is identical with a tangent space to a certain point on two-dimensional sphere.

Step 2: Lorentzian metric on $\mathcal{P}$.

A Lorentzian metric $\tilde{g}$ can be defined on $\mathcal{P}$ by the requirements that:

(i) a scalar product of any two horizontal vectors in $\tilde{g}$ is the same as the scalar product in g of their push forwards to $\mathcal{M}$,

(ii) a scalar product of any two vertical vectors in $\tilde{g}$ is equal to their scalar product in a natural metric on two-dimensional sphere (this is consistent since vertical vectors can be considered tangent vectors to $\mathbf{S}^2$),

(iii) any two vectors such that one is horizontal and the other is vertical are orthogonal in $\tilde{g}$.

If w_i $(i = 1, 2)$ are two vectors at a point $p \in \mathcal{P}$, v_i, h_i denote their vertical and horizontal components, respectively and Σ denotes a natural metric on $\mathbf{S}^2$, then definition of $\tilde{g}$ can be written more precisely as

$$\tilde{g}(w_1, w_2) = g(\pi_*(h_1), \pi_*(h_2)) + \Sigma(v_1, v_2).$$

Step 3: Null spray on $\mathcal{P}$.

A natural congruence on $\mathcal{P}$ is related to the horizontal lifts of null directions from $\mathcal{M}$. It is defined by the following recipe. Take a null vector k at $x \in \mathcal{M}$. This represents certain null direction $p(k)$ outgoing from x. Correspondingly, this defines a point $p = p(k)$ in the fiber $\pi^{-1}(x)$. Lift k horizontally to p. This defines $\tilde{k}$ which generates a certain direction outgoing from $p \in \mathcal{P}$. Repeating this procedure for all directions outgoing from $x \in \mathcal{M}$ we attach to any point of $\pi^{-1}(x)$ a unique direction. If we do it for all points of $\mathcal{M}$, we define a field of directions on $\mathcal{P}$ which, according to its construction and properties of $\tilde{g}$, is null. Integral curves of this field form a null congruence required in point (1) of definition of optical geometry. This congruence is called null spray on $\mathcal{P}$ [34].

Step 4: Almost complex structure in screen space.

In any point $p \in \mathcal{P}$ tangent vectors to the null spray span one-dimensional vector space K_p, which is null with respect to $\tilde{g}$. In this way null spray defines a vector bundle $\mathcal{K} = \bigcup_{p\in\mathcal{P}} K_p$ with null fibers of dimension 1. A bundle orthogonal to it in $\tilde{g}$ is denoted by $\mathcal{K}^{\perp}$. This is of fiber dimension 5, and has $\mathcal{K}$ as its subbundle. A quotient bundle $\mathcal{H} = \mathcal{K}^{\perp}/\mathcal{K}$ has fibers of dimension 4 and is called a screen space for the null spray. Since $\tilde{g}$ is degenerate on $\mathcal{K}$ then it descends to a Euclidean metric $\tilde{g}'$ in $\mathcal{H}$. To complete the definition of optical geometry on $\mathcal{P}$ we need to define such an almost complex structure $\mathcal{J}$ in $\mathcal{H}$, that it is orthogonal with respect to $\tilde{g}'$.

Let k denotes a null vector at $x \in \mathcal{M}$. Consider a subspace $k^\perp$ of $\mathrm{T}_x\mathcal{M}$ which in g is orthogonal to k. A space $\mathbb{R}k$ spanned in $\mathrm{T}_x\mathcal{M}$ by k is its subspace. Hence we can define vector space $\chi = k^\perp/\mathbb{R}k$. This two-dimensional space is naturally endowed with an orientation induced from that of $\mathcal{M}$ and with the metric g' induced by g. An orientation-clockwise rotation of any vector from χ to a vector orthogonal to it defines a complex structure j_x in χ. Horizontal lifts of χ and j_x to the point $p = p(k) \in \mathcal{P}$ are defined as follows. Denote respectively by $\widetilde{k^\perp}$ and $\widetilde{\mathbb{R}k}$ spaces $k^\perp$ and $\mathbb{R}k$ lifted horizontally to the point p. A horizontal lift $\tilde{\chi}_p$ of χ is $\tilde{\chi}_p = \widetilde{k^\perp}/\widetilde{\mathbb{R}k}$. Spaces χ and $\tilde{\chi}_p$ are naturally isomorphic, and one can use j_x and this isomorphism to define a complex structure j_p in $\tilde{\chi}_p$. Since $\widetilde{\mathbb{R}k}$ is precisely a direction of null spray at p, and since $\widetilde{k^\perp}$ is a subspace of the fiber $K_p^\perp$ of $\mathcal{K}^\perp$, then we see that $\tilde{\chi}_p$ is a subspace of a fiber H_p of bundle $\mathcal{H}$. To identify the complement of $\tilde{\chi}_p$ in H_p we note, that V_p is always orthogonal to null spray and, being vertical, does not contain K_p. Therefore it descends to a two-dimensional subspace (also denoted by V_p) in the fiber H_p of the quotient bundle $\mathcal{H}$. Now, it is easy to see that so defined V_p is a complement of $\tilde{\chi}_p$ in H_p. Moreover, these two spaces are orthogonal in the metric $\tilde{g}'$. Remembering that V_p can be identified with the tangent space of the two-dimensional sphere, we endow it with the complex structure i_p, that comes from the natural complex structure on $\mathbf{S}^2$. In this way, we arrived at the split $H_p = V_p \bigoplus \tilde{\chi}_p$ and we have well-defined complex structures j_p in $\tilde{\chi}_p$ and i_p in V_p. This gives two possibilities of defining complex structures in H_p. These are $J_{+p} = i_p + j_p$ and $J_{-p} = i_p - j_p$, where addition is understood as a direct sum. This, point by point, defines two almost complex structures $\mathcal{J}_+$ and $\mathcal{J}_-$ in the bundle $\mathcal{H}$. It is easy to check that both structures are orthogonal with respect to the metric $\tilde{g}'$.

Summing up, we defined two natural structures $\mathcal{O}_+ = (\mathcal{K}, \tilde{g}, \mathcal{J}_+)$ and $\mathcal{O}_- = (\mathcal{K}, \tilde{g}, \mathcal{J}_-)$ of optical geometries on $\mathcal{P}$. One can view them as the following sequence:

$$\begin{array}{lccc} & \mathcal{K} \hookrightarrow & \mathcal{K}^\perp \longrightarrow & \mathcal{H} = \mathcal{K}^\perp/\mathcal{K} \\ \text{fiber dimension} & 1 & 5 & 4 \end{array}$$

of real vector bundles, and differ them stating which of two natural orthogonal almost complex structures $\mathcal{J}_+$ or $\mathcal{J}_-$ is given in $\mathcal{H}$.

3. Optical geometries and CR-structures. General theory

We ended the last section with two natural optical geometries $\mathcal{O}_\pm$ on $\mathcal{P}$. Although they differ only by an almost complex structure in the bundle $\mathcal{H}$, their properties are totally different. To see this difference we return to the general case of $2m$-dimensional manifold $\mathcal{X}$ with Lorentzian metric g. An optical geometry $\mathcal{O}$ on it is defined by:

(1) the following sequence

$$\begin{array}{lccc} & \mathcal{K} \hookrightarrow & \mathcal{L} = \mathcal{K}^\perp \longrightarrow & \mathcal{H} = \mathcal{L}/\mathcal{K} \\ \text{fiber dimension} & 1 & 2m-1 & 2m-2 \end{array}$$

of real vector bundles among which $\mathcal{K}$ is a null subbundle of $T\mathcal{X}$, and

(2) an almost complex structure $\mathcal{J}$ in $\mathcal{H}$ which is orthogonal with respect to the metric g' induced by g in $\mathcal{H}$.

Suppose now that we have an optical geometry $\mathcal{O}$ that satisfies the following conditions:

(a) $[\Gamma(\mathcal{K}), \Gamma(\mathcal{K})] \subset \Gamma(\mathcal{K})$,

(b) $[\Gamma(\mathcal{K}), \Gamma(\mathcal{L})] \subset \Gamma(\mathcal{L})$,

(c) *if ϕ_k is a flow generated by any section k of $\mathcal{K}$ then $\tilde{\phi}_k \mathcal{J} = \mathcal{J}$.*

Condition (a) is always satisfied. It says that $\Gamma(\mathcal{K})$ defines a foliation of $\mathcal{X}$ by one-dimensional manifolds – integral curves of any nonvanishing section $k \in \Gamma(\mathcal{K})$. These form a null congruence on $\mathcal{X}$. Condition (b) says that any line of this congruence is geodesic in the metric g. Any point $x \in \mathcal{X}$ belongs to precisely one line of the congruence. We define an equivalence relation "$\sim$" on $\mathcal{X}$ identifying points on the same line. More precisely, any two points $x, x' \in \mathcal{X}$ are in the relation "$\sim$" iff $x' = \phi_k(x)$.

We assume that in a considered region $\mathcal{U}$ of $\mathcal{X}$ a quotient space $\mathcal{Q} \stackrel{\text{def}}{=} \mathcal{U}/\sim$ is a $(2m-1)$-dimensional manifold. This quotient manifold has some additional structure. Let Π denote the canonical projection $\Pi : \mathcal{U} \to \mathcal{U}/\sim$. If (a)–(c) holds then this projection defines $\mathcal{H}' \stackrel{\text{def}}{=} \Pi_* \mathcal{H}$ and $\mathcal{J}' \stackrel{\text{def}}{=} \tilde{\Pi}\mathcal{J}$. One checks that $\mathcal{H}'$ is a subbundle of $T\mathcal{Q}$ with fibers of dimension $2(m-1)$ and $\mathcal{J}'$ is an almost complex structure in $\mathcal{H}'$. Now, let X', Y' be any two real sections of the bundle $\mathcal{H}'$. Note that X', Y' are now vector fields on $\mathcal{Q}$. Extending $\mathcal{J}'$ to $\mathbb{C} \otimes \mathcal{H}'$ by linearity one can ask when

(d) $\mathcal{J}'[X' + \mathrm{i}\mathcal{J}'X',\ Y' + \mathrm{i}\mathcal{J}'Y'] = -\mathrm{i}[X' + \mathrm{i}\mathcal{J}'X',\ Y' + \mathrm{i}\mathcal{J}'Y']$.

This condition generalizes classical Newlander–Nirenberg integrability conditions [19] of an almost complex structure, in the sense that if $\mathcal{H}'$ defines a foliation of $\mathcal{Q}$ by $(2m-2)$-dimensional manifolds then $\mathcal{J}'$ restricted to these manifolds and satisfying (d) is a complex structure there.

We say that an optical geometry $\mathcal{O}$ is integrable if and only if it satisfies the above conditions (a)–(d). For obvious reasons, optical geometry satisfying only conditions (a)–(b) is called geodesic.

There is a relation between integrable optical geometries and CR-structures.

A Cauchy–Riemann (CR) structure is a real $(2m-1)$-dimensional manifold $\mathcal{Q}$ together with:

(1) a real subbundle $\mathcal{H}' \subset T\mathcal{Q}$ of fibers of dimension $2(m-1)$,

(2) an almost complex structure $\mathcal{J}'$ in $\mathcal{H}'$.

If in addition CR-structure satisfies integrability condition (d) then it is called an integrable CR-structure. Equivalently, CR-structure may be also defined on $\mathcal{Q}$ by $2m$ complex valued 1-forms E_i $(i = 1, 2, \ldots, 2m)$, and one real valued 1-form Λ such that

$$\Lambda \wedge E_1 \wedge E_2 \wedge \cdots \wedge E_{2m} \wedge \bar{E}_1 \wedge \bar{E}_2 \wedge \cdots \wedge \bar{E}_{2m} \neq 0. \tag{4}$$

Then $\mathcal{H}'$ is a vector bundle whose all sections are complex valued vector fields on $\mathcal{Q}$ that annihilate Λ and all E_i's. So defined $\mathcal{H}'$ is naturally endowed with an almost complex structure. Note that forms

$$\Lambda' = f\Lambda \quad \text{and} \quad E'_i = p_i^{\,j} E_j, \tag{5}$$

with real function $f \neq 0$ and complex functions p_i^j, $p_i^i \neq 0$ define the same $\mathcal{H}'$. Therefore forms (Λ, E_i) and (Λ', E_i') define the same CR-structure. This shows why one also defines a CR-structure on $\mathcal{Q}$ in terms of a *class* of 1-forms $[(\Lambda, E_i)], i = 1, ..., 2m$ given up to transformations $(\Lambda, E_i) \to (\Lambda', E_i')$.

Two CR-structures $(\mathcal{Q}, \mathcal{H}, \mathcal{J})$ and $(\mathcal{Q}', \mathcal{H}', \mathcal{J}')$ are called (locally) equivalent iff there exists a (local) diffeomorphism $\phi : \mathcal{Q} \to \mathcal{Q}'$ such that

$$\phi_* \mathcal{H} = \mathcal{H}' \quad \text{and} \quad \tilde{\phi} \mathcal{J} = \mathcal{J}'.$$

There exist nonequivalent CR-structures on the same manifold $\mathcal{Q}$.

We have the following theorem.

Theorem 3.1.
(1) *A real 2m-dimensional manifold $\mathcal{X}$ equipped with an integrable optical geometry is locally diffeomorphic to $\mathbb{R} \times \mathcal{Q}$ where $\mathcal{Q}$ is a $(2m-1)$-dimensional integrable CR-structure.*
(2) *Given a $(2m-1)$-dimensional integrable CR-structure $\mathcal{Q}$ one can define an integrable optical geometry on $\mathbb{R} \times \mathcal{Q}$.*

Proof. Point (1) is obvious in view of what we have said so far. The local factor $\mathbb{R}$ in $\mathbb{R} \times \mathcal{Q}$ is associated with integral curves of any nonvanishing section k of $\mathcal{K}$. We prove (2) by giving a construction of an integrable optical geometry. Let Π denote the projection $\Pi : \mathbb{R} \times \mathcal{Q} \to \mathcal{Q}$. Denote by $\mathcal{S}$ a subbundle of $T^*\mathcal{Q}$ annihilating $\mathcal{H}'$. $\mathcal{S}$ has one-dimensional fibers, hence it is generated by a nonvanishing section, say λ'. We define a nonvanishing 1-form λ on $\mathbb{R} \times \mathcal{Q}$ by $\lambda = \Pi^*(\lambda')$. We define k as a nonvanishing vector field on $\mathbb{R} \times \mathcal{Q}$ which is tangent to the fibers $\Pi^{-1}(x)$, $x \in \mathcal{Q}$. We define $\mathcal{K}$ as a bundle generated by k, and $\mathcal{L} \stackrel{\text{def}}{=} \ker \lambda$; both $\mathcal{K}$ and $\mathcal{L}$ are subbundles of $T(\mathbb{R} \times \mathcal{Q})$. Due to the construction $\Pi^*(\mathcal{L}/\mathcal{K}) = \mathcal{H}'$. Therefore we can define $\mathcal{J}$ in $\mathcal{H} = \mathcal{L}/\mathcal{K}$ by $\mathcal{J}([l]) = \mathcal{J}'(\Pi_*([l]))$, where $[l] \in \mathcal{L}/\mathcal{K}$. Let g' be any Euclidean metric in $\mathcal{H}'$ such that $\mathcal{J}'$ is orthogonal in it. We define a Lorentzian metric on $\mathbb{R} \times \mathcal{Q}$ by

$$g = \Pi^*(g') - \lambda\alpha,$$

where α is any 1-form on $\mathbb{R} \times \mathcal{Q}$ such that the metric g is nondegenerate. One checks that $(\mathcal{K}, g, \mathcal{J})$ is an integrable optical geometry with g as a metric on $\mathbb{R} \times \mathcal{Q}$. □

For $m = 2$ the above theorem was formulated in [29]. In somewhat different context it also appeared in [18,32,33,35].

Note that the metric g in the proof of Theorem 3.1 is not uniquely defined. It follows from this proof that an optical geometry admits a whole class of the so-called adapted metrics. A Lorentzian metric G on $\mathcal{X}$ is said to be adapted to the optical geometry $(\mathcal{K}, g, \mathcal{J})$ if, relative to G:
(I) $\mathcal{K}$ is null,
(II) $\mathcal{K}^\perp$ is orthogonal to $\mathcal{K}$,
(III) $\mathcal{J}$ is orthogonal with respect to metric G' induced by G in $\mathcal{H}$.

We note that the case $m = 2$ is different from cases $m > 2$ in the following sense.

If $m = 2$ and $\mathcal{X}$ is oriented then any optical geometry can be defined by choosing a null congruence on $\mathcal{X}$. All vector fields tangent to the congruence form the bundle $\mathcal{K}$ and due to the dimension of $\mathcal{X}$, the bundle $\mathcal{H} = \mathcal{K}^{\perp}/\mathcal{K}$ has two-dimensional fibers. Then the orientation of $\mathcal{X}$ and the metric g' in $\mathcal{H}$ uniquely define a complex structure $\mathcal{J}$ in $\mathcal{H}$. The integrability conditions (a)–(d) for so obtained optical geometry are equivalent to the geodesic and shear-free property of the congruence [29,30]. If they are satisfied in one of the adapted metrics they are also valid in any other adapted metric.

This is not in general true if $m > 2$. In such cases a null congruence also defines $\mathcal{K}$ and $\mathcal{K}^{\perp}$ as before but, since the fiber dimension of $\mathcal{H} = \mathcal{K}^{\perp}/\mathcal{K}$ is greater than or equal to 4, there is no natural way of defining $\mathcal{J}$ in $\mathcal{H}$. Therefore if $m > 2$ the choice of a null congruence does not suffice for a definition of an optical geometry. Moreover, the integrability conditions (a)–(d) say nothing about the shear-free property of the congruence generated by sections of the bundle $\mathcal{K}$. It may happen that in some adapted metrics the congruence has shear and in some others not [28].

4. Integrability conditions for $\mathcal{O}_{\pm}$ on $\mathcal{P}$

In Section 3 we mentioned that two natural optical geometries $\mathcal{O}_{+}$ and $\mathcal{O}_{-}$ on twistor bundle $\mathcal{P}$ are different. To be more precise we have the following theorem.

Theorem 4.1.
(1) *Both optical geometries $\mathcal{O}_{\pm}$ are geodesic.*
(2) *$\mathcal{O}_{-}$ is never integrable.*
(3) *$\mathcal{O}_{+}$ is integrable if and only if the metric g on base manifold $\mathcal{M}$ is conformally flat.*

Sketch of the proof. One checks integrability conditions (a)–(d) for optical geometries $\mathcal{O}_{\pm}$ by a straightforward calculation. Remarkably, one finds that in addition to a trivially satisfied condition (a) also condition (b) is satisfied automatically. Since these two conditions are independent of the choice of $\mathcal{J}$ in $\mathcal{H}$ then this proves point (1) of the theorem. Note that this, in particular, means that the null spray on $\mathcal{P}$ is geodesic. This justifies a name 'null geodesic spray', that some people use instead of null spray [34].

Inspecting condition (c) one finds the main difference between optical geometries $\mathcal{O}_{+}$ and $\mathcal{O}_{-}$. It turns out that $\mathcal{O}_{-}$ never satisfies condition (c). This means that independently of the properties of the metric g on $\mathcal{M}$ geometry $\mathcal{O}_{-}$ is not integrable. On the other hand one finds that $\mathcal{O}_{+}$ satisfy condition (c) if and only if the metric g on $\mathcal{M}$ is conformally flat. To be more specific, we parametrize any point $p \in \mathcal{P}$ by (x, z), where $x = \pi(p)$ and z is a number from the Argand plane as defined in Section 2. Then one finds that condition (c) is satisfied if and only if the following expression

$$\Psi_0(z) = \Psi_0 + 4z\Psi_1 + 6z^2\Psi_2 + 4z^3\Psi_3 + z^4\Psi_4$$

vanishes for any z. This means that all Weyl scalars Ψ_i $(i = 0, 1, 2, 3, 4)$ of the metric g vanish. Hence g must be conformally flat.

Now, restricting ourselves to conformally flat metrics on $\mathcal{M}$ we find quite unexpectedly that condition (d) for $\mathcal{O}_+$ associated with such metrics is satisfied automatically. This completes the proof. □

It follows from the sketch of the proof that the whole set of integrability conditions (a)–(d) for the optical geometry $\mathcal{O}_+$ is equivalent to the only one identity $\Psi_0(z) \equiv 0$. One can also ask about those z's for which we have $\Psi_0(z) = 0$. There are at most four such z's. These are precisely the same that via (2) correspond to principal null directions on $\mathcal{M}$. They define special sections $z : \mathcal{M} \ni x \mapsto (x, z(x)) \in \mathcal{P}$ of the twistor bundle on which conditions (a)–(d) are satisfied. The set of all such sections is called a set of zeroes of integrability conditions for $\mathcal{O}_+$. It consists of at most four elements. Due to our identification of sections of $\mathcal{P}$ with fields of null directions on $\mathcal{M}$ we have the following theorem.

Theorem 4.2. *Principal null directions on a four-dimensional Lorentzian manifold are identical with the set of zeroes of integrability conditions for optical geometry $\mathcal{O}_+$.*

This theorem can be considered a geometrical definition of principal null directions.

5. Encoding traceless part of the Einstein equations on $\mathcal{P}$

The following reformulation of integrability conditions (a)–(d) for optical geometry $\mathcal{O}_-$ is possible.

Fix a metric g on the base manifold $\mathcal{M}$ of $\mathcal{P}$. Let N be any nonvanishing vector field tangent to the null spray on $\mathcal{P}$. Let Λ be a real one form on $\mathcal{P}$ defined by $\Lambda = \tilde{g}(N)$. It is specified up to a multiplication by a real function on $\mathcal{P}$ ($\Lambda \to \Lambda' \neq 0$ s.t. $\Lambda' \wedge \Lambda = 0$). With the horizontal space in $\mathcal{P}$ one associates another 1-form. This is such a complex 1-form E_1 on $\mathcal{P}$ that (i) it annihilates horizontal space and (ii) $E_1 \wedge \bar{E}_1 \neq 0$. This is also defined up to a multiplication by a complex function on $\mathcal{P}$ ($E_1 \to E_1' \neq 0$ s.t. $E_1' \wedge E_1 = 0$). It is easy to see that the metric $\tilde{g}$ on $\mathcal{P}$ can be expressed as

$$\tilde{g} = 2(a^2 E_1 \bar{E}_1 + \Lambda S + b^2 E_2 \bar{E}_2)$$

with some nonvanishing functions a, b and some 1-forms S (real) and E_2 (complex) on $\mathcal{P}$. The above expression can be considered a definition of the form E_2. It is given up to such transformations $E_2 \to E_2'$ that $E_2' \wedge E_2 \wedge \Lambda = 0$ and $E_2' \wedge \Lambda \neq 0$. One can check that the following theorem is true.

Theorem 5.1. *Optical geometry $\mathcal{O}_+$ on $\mathcal{P}$ is integrable iff*

$$(\mathcal{L}_N E_1) \wedge E_1 \wedge E_2 \wedge \Lambda = 0. \qquad (6)$$

The proof of this theorem consists of simple calculations. Here we mention only that integrability condition (b) of $\mathcal{O}_+$ is equivalent to $(\mathcal{L}_N \Lambda) \wedge \Lambda = 0$ and is, of course, satisfied trivially. We also know that the only condition to be satisfied is (c). One checks that it splits into automatically satisfied part

$$(\mathcal{L}_N E_2) \wedge E_1 \wedge E_2 \wedge \Lambda = 0, \tag{7}$$

and condition (6), which is satisfied if and only if the metric g of the base manifold $\mathcal{M}$ is conformally flat.

The fact that integrability conditions for $\mathcal{O}_+$ imply conformal flatness of the base metric g is discouraging. One would rather like to have these kind of conditions to code Einstein equations for g. That this is not so hopeless shows the following theorem.

Theorem 5.2. *The following condition*

$$(\mathcal{L}_N E_1) \wedge E_1 \wedge \bar{E}_2 \wedge \Lambda = 0 \tag{8}$$

is equivalent to the Einstein equations

$$R_{\mu\nu} = \lambda g_{\mu\nu} \tag{9}$$

with cosmological constant λ.

Our proof of this theorem is by direct calculations. We mention only that condition (8) is equivalent to condition

$$\begin{aligned}\Phi_{00}(z) = {} & \Phi_{00} + 2\bar{z}\Phi_{01} + 2z\bar{\Phi}_{01} + 4z\bar{z}\Phi_{11} + \bar{z}^2\Phi_{02} + z^2\bar{\Phi}_{02} \\ & + 2\bar{z}^2 z\Phi_{12} + 2z^2\bar{z}\bar{\Phi}_{12} + z^2\bar{z}^2\Phi_{22} \equiv 0,\end{aligned} \tag{10}$$

which must be satisfied identically for all z's. Details are presented in [21].

Comparing conditions (6) and (8) one sees that they differ only by interchanging E_2 and $\bar{E}_2$. That this symmetry can be misleading follows from the fact, that one has to remember about condition (7), if one wants to interpret (8) as integrability conditions for some other optical geometry, say $\mathcal{O}'$. It is interesting to ask whether there exists some geometrical structure on $\mathcal{P}$ for which (8) can be interpreted as integrability condition.

Theorems 5.1 and 5.2 can be also interpreted as follows. Suppose that $\mathcal{S}$ is any surface of dimension 5 in $\mathcal{P}$ that is transversal to the null geodesic spray N. Consider the restrictions $\Lambda(\mathcal{S})$, $E_1(\mathcal{S})$, $E_2(\mathcal{S})$ of forms Λ, E_1, E_2 to $\mathcal{S}$, respectively. These forms generate a CR-structure $\mathcal{Q}(\mathcal{S})$ on $\mathcal{S}$. Let $\mathcal{Q}(\mathcal{S}')$ be a transport of $\mathcal{Q}(\mathcal{S})$ along null geodesic spray from $\mathcal{S}$ to another transversal surface $\mathcal{S}'$. A typical situation will be such that two CR-structures $\mathcal{Q}(\mathcal{S})$ and $\mathcal{Q}(\mathcal{S}')$ will be inequivalent. According to Theorems 5.1 and 5.2 the knowledge of a deformation of CR-structure $\mathcal{Q}(\mathcal{S})$ during its transport along N carries an information about properties of the metric g on the base $\mathcal{M}$ of the twistor bundle. This, in particular, gives a possibility of constructing Lorentzian Einstein manifolds. One could first try to construct a six-dimensional real manifold via 1-parameter deformation $(\Lambda(r), E_1(r), E_2(r))$ of a given five-dimensional CR-structure (Λ, E_1, E_2). This deformation should be subject to the following equations:

$$(\mathcal{L}_N \Lambda(r)) \wedge \Lambda(r) = 0, \tag{11}$$

$$(\mathcal{L}_N E_2(r)) \wedge E_2(r) \wedge E_1(r) \wedge \Lambda(r) = 0, \tag{12}$$

$$(\mathcal{L}_N E_1(r)) \wedge \bar{E}_2(r) \wedge E_1(r) \wedge \Lambda(r) = 0, \tag{13}$$

where $N = \partial_r$. Given such a deformation, one could try to identify it with a twistor space of some four-dimensional Lorentzian manifold. If such procedure of identification exists, it should then allow for reconstruction of the four-dimensional Lorentzian metric that satisfies Einstein equations.

We close this section with the following remark. If one asks about sections of the twistor bundle on which condition (8) vanishes one finds that they correspond to the principal null directions of the traceless Ricci tensor. (Note similarity between $\Psi_0(z) = 0$ and $\Phi_{00}(z) = 0$.)

6. Optical and Hermitian geometries – an analogy

There is a striking analogy between optical and Hermitian geometries [9,38]. To describe this we consider a general $2m$-dimensional manifold $\mathcal{X}$ with Lorentzian or Euclidean metric g. In the complexification of $\mathrm{T}\mathcal{X}$, equipped with the complexification of the real metric g, consider such a subbundle $\mathcal{N}$, that any of its fibers is a totally null vector space of maximal dimension. It turns out that $\mathcal{N} \cap \bar{\mathcal{N}} = \mathbb{C} \otimes \mathcal{K}$ where $\mathcal{K}$ is a real bundle of fiber dimension 0 or 1, depending on whether g is Euclidean or Lorentzian, respectively. Moreover, since $\mathcal{N} + \bar{\mathcal{N}} = \mathbb{C} \otimes \mathcal{K}^\perp$ then $\mathcal{L} = \mathcal{K}^\perp$ is a subbundle of $\mathrm{T}\mathcal{X}$ with fibers of codimension 0 (in the Euclidean case) or 1 (in the Lorentzian case). In both cases we have a natural almost complex structure $\mathcal{J}$ in $\mathcal{H} \stackrel{\text{def}}{=} \mathcal{L}/\mathcal{K}$. To define this we observe that any section l of $\mathcal{L}$ is of the form $l = n + \bar{n}$ where n is some section of $\mathcal{N}$. If $[l]$ denotes an equivalence class associated with l in $\mathcal{H}$ we define $\mathcal{J}$ by

$$\mathcal{J}(\,[l]\,) = \mathcal{J}(\,[n + \bar{n}]\,) \stackrel{\text{def}}{=} [-\mathrm{i}\,(n - \bar{n})\,]. \tag{14}$$

One proves that $\mathcal{J}$ is well defined and orthogonal with respect to the descended metric g' in $\mathcal{H}$. Therefore $\mathcal{N}$ defines either almost Hermitian or optical geometry over $\mathcal{X}$, depending on whether the signature of g is Euclidean or Lorentzian, respectively. The converse is also true. Any orthogonal almost complex structure or optical geometry over $\mathcal{X}$ can be obtained in this manner.

Thus we have an *analogy* between almost complex and almost optical geometries. Due to this, integrability conditions (a)–(d) for optical geometries and classical Newlander–Nirenberg integrability conditions for Hermitian geometries have a uniform description. These are equivalent [38] to

$$[\Gamma(\mathcal{N}), \Gamma(\mathcal{N})] \subset \Gamma(\mathcal{N}). \tag{15}$$

The above analogy leads to the following programme of Trautman. Find all optical geometric counterparts of known theorems and constructions in Hermitian geometry. The same programme can be also applied in the opposite direction. We have only partial knowledge in this matter. We know, in particular, that $\mathcal{O}_+$ on $\mathcal{P}$ is analogous to the Atiyah–Hitchin–Singer construction [1], of a natural almost Hermitian structure on the space $\mathcal{A}$ of all almost Hermitian structures over four-dimensional Euclidean manifold. [2] A fact that well known to relativists Goldberg–Sachs theorem [8] has a Hermitian counterpart [27] is somewhat surprising to us. [3] This result states necessary and sufficient conditions for local existence of integrable Hermitian structure on four-dimensional Euclidean manifolds satisfying the Einstein equations (9). Such structures can exist iff the metric of the manifold is algebraically special, in the sense of the Euclidean analog of Petrov classification (see also [20]).

More facts concerning the analogy between optical and Hermitian structures are given below. In the following list $\mathcal{M}$ is an oriented, Lorentzian or Euclidean, respectively, four-dimensional manifold.

Optical geometry – Hermitian geometry

(1) Null vector field k on $\mathcal{M}$ – almost Hermitian structure J on $\mathcal{M}$.
(2) Two spheres of null directions outgoing from or ingoing to $x \in \mathcal{M}$ – two spheres of almost Hermitian structures at $x \in \mathcal{M}$.
(3) Shear-free and geodesic property for k – integrability for J.
(4) $\mathcal{P} - \mathcal{A}$.
(5) $\mathcal{O}_+$ – standard almost Hermitian structure $\mathcal{J}_+$ on $\mathcal{A}$.
(6) $\mathcal{O}_-$ – nonstandard almost Hermitian structure $\mathcal{J}_-$ on $\mathcal{A}$.
(7) Theorem 4.1 – theorem of Ref. [1] on integrability of $\mathcal{J}_\pm$.
(8) Principal null directions of Cartan [4] – sections on which integrability conditions for $\mathcal{J}_+$ vanish.
(9) Kerr theorem – which Hermitian structures live in self-dual $\mathcal{M}$.
(10) Goldberg–Sachs theorem [8] – Przanowski–Broda theorem [27].
(11) Degenerate points of Rovelli–Smolin loop variables [7] – Kahlerian 4-manifolds.
(12) Integrability of CR-structure associated with congruence of twisting shear-free and null geodesics on Einstein $\mathcal{M}$ [17] – existence of holomorphic vector field on Einstein algebraically special conformally nonflat and not Kahlerian $\mathcal{M}$ [5], [26]. [4]
(13) ? – construction of instantons [1].
(14) ? – Salamon's construction [31] of harmonic maps using $\mathcal{J}_-$.
(15) Fefferman metric [6] for three-dimensional CR-structure – some self-dual class of four-dimensional Euclidean metrics (?).

One can continue this list. The following remarks are in order.

[2] In Ref. [1] space $\mathcal{A}$ is called twistor bundle. It is the reason why we also assign this name to $\mathcal{P}$. Relativist twistorians prefer a name 'spin bundle' for $\mathcal{P}$.

[3] For example, up to our knowledge it is not noted in Besse's book [2].

[4] I thank Lewandowski for bringing this to my attention. See also [16].

(A) According to Lewandowski [15] our formulation (8) of the traceless part of the Einstein equations in terms of distinguished forms on $\mathcal{P}$ seems to have no Euclidean counterpart.

(B) Kobak [10] asks as to whether one can use $\mathcal{O}_-$ to find an analog of Salamon's harmonic map construction. In Salamon's paper [31] a crucial role was played by the almost Hermitian structure $\mathcal{J}_-$ on $\mathcal{A}$. This suggests that also $\mathcal{O}_-$ can be useful for solving certain nonlinear equations on Lorentzian 4-manifolds.

(C) For any nondgenerate three-dimensional CR-structure $\mathcal{Q}$ we can define the Fefferman conformal class of Lorentzian metrics on the fiber bundle over $\mathcal{Q}$. This class of metrics is always of Petrov type N [13]. All Fefferman metrics can be also defined as one of the two subclasses of four-dimensional Lorentzian metrics that admit nonzero solution to the twistor equation [5] [14]. Euclidean metrics that admit solutions to the twistor equation must be self-dual [15]. Does there exist a Euclidean analog of the Fefferman [6] (or, more likely, of the Burns, Diederich and Schnider [3]) construction of Fefferman metrics? If yes, does it provide us with a new class of self-dual Hermitian metrics?

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